CHAPTER VII.

THE HYPERBOLA.

99. The **Hyperbola** is the locus of a point moving in a plane so that its distance from a fixed point bears a constant ratio to its distance from a fixed right line, the ratio being greater than unity.*

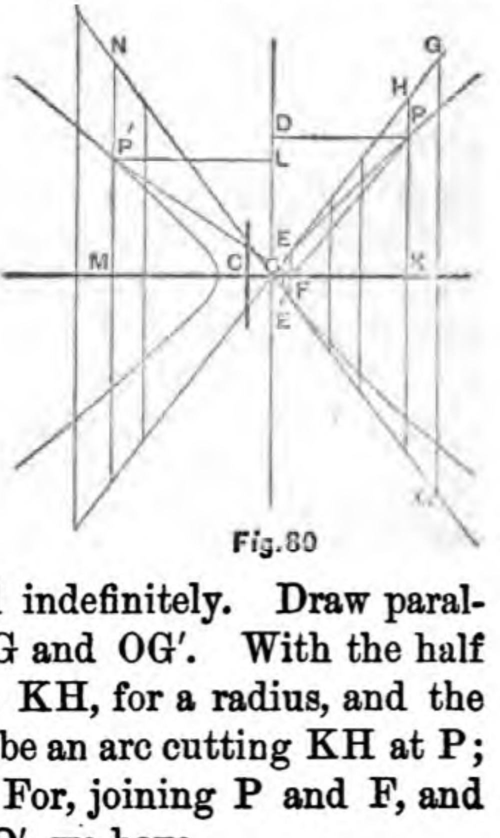

Fig. 80

From this definition the hyperbola may be constructed by points, thus:

Let F be the fixed point, DD′ the fixed right line, and *e* the given ratio. Draw through F the line OAF perpendicular and EE′ parallel to DD′. Take

$$FE\ (= FE') : FO :: e : 1,$$

and draw OE and OE′ produced indefinitely. Draw parallels to EE′, meeting the lines OG and OG′. With the half of any one of these parallels, as KH, for a radius, and the fixed point F for a centre, describe an arc cutting KH at P; this is a point of the curve. For, joining P and F, and drawing PD perpendicular to DD′, we have

$$KH\ (= FP) : KO\ (= PD)\ ::\ FE : FO;$$

that is, by construction we have

$$FP : PD :: e : 1.$$

In the same way, any required number of points in the curve may be found.

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* See Todhunter's Conic Sections, p. 188.

Since $e > 1$, the distance from F to any point in the curve is greater than the distance from the same point to the line DD′; therefore there are points in the curve on the opposite side of DD′, which are found in the same way as those to the right of DD′, thus: with the half of any of the parallels on the left of DD′, as MN, for a radius, and F for a centre, describe an arc cutting MN at P′; this is a point of the curve. For, joining P′ and F, and drawing P′L perpendicular to DD′, we have

$$MN\ (= FP') : MO\ (= P'L)\ ::\ FE : FO;$$

that is, by construction we have $FP' : P'L\ ::\ e : 1$.

In the same way, any required number of points may be found. Connecting these points by a line, we have the required hyperbola.

The fixed line DD′ is called the **Directrix**; the fixed point F is called the **Focus**; OG and OG′ are called the **Focal Tangents**.

100. *To find the equation of the hyperbola.*

Let F be the focus, and YY′ the directrix; through F draw OX perpendicular to YY′; take OX and OY for the co-ordinate axes. Let (x, y) be any point P on the curve; join FP; draw PM and PD respectively perpendicular to OX and OY.

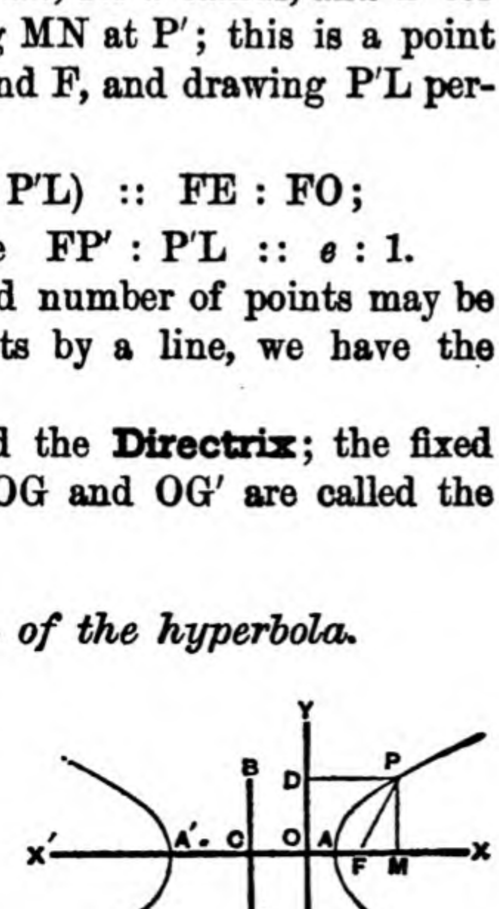

Fig. 81.

Represent OF by p, and the ratio of FP to PD by e. Then we have, from definition,

$$FP = e \cdot PD, \quad \text{or} \quad \overline{FP}^2 = e^2 \cdot \overline{PD}^2;$$

or $$\overline{FM}^2 + \overline{MP}^2 = e^2 \cdot \overline{PD}^2;$$

that is, $$(x - p)^2 + y^2 = e^2x^2, \qquad (1)$$

which is the required equation.

COR.—When $y = 0$, we have $x - p = \pm ex$; therefore,

$$x = \frac{p}{1 \mp e},$$

which shows that the curve cuts the axis of x at the two points A and A′, giving

$$\text{OA} = \frac{p}{1+e}, \qquad \text{and} \qquad \text{A}'\text{O} = -x = \frac{p}{e-1}.$$

Since $e > 1$, $1 - e$ is a negative quantity; therefore OA′ must be measured to the *left* of the origin.

A and A′ are called the **Vertices** of the hyperbola, and C, the point midway between them, is called the **Centre** of the hyperbola.

$$\text{The distance AA}' = \text{A}'\text{O} + \text{OA}$$

$$= \frac{p}{e-1} + \frac{p}{1+e} = \frac{2ep}{e^2-1};$$

it is customary to denote this distance by $2a$. Therefore we have

$$2a = \frac{2ep}{e^2-1}.$$

101. Transform equation (1) of Art. 100 to A (Fig. 81).

The formulæ for this transformation become (Art. 100, Cor.),

$$x = \frac{p}{1+e} + x', \qquad y = y',$$

which in (1) gives

$$\left(\frac{p}{1+e} + x' - p\right)^2 + y'^2 = e^2\left(\frac{p}{1+e} + x'\right)^2,$$

or
$$x'^2 + y'^2 - 2\frac{ep}{1+e}x' = e^2\left(\frac{2p}{1+e}x' + x'^2\right);$$

therefore,
$$y'^2 = 2epx' - (1 - e^2)x'^2$$

$$= (1 - e^2)\left(\frac{2ep}{1-e^2}x' - x'^2\right);$$

therefore (Art. 100, Cor.),

$$y'^2 = (e^2 - 1)(2ax' + x'^2),$$

or, suppressing the accents, since the variables are general, we have

$$y^2 = (e^2 - 1)(2ax + x^2). \qquad (1)$$

Cor.—When $x = -a$, (1) becomes

$$y^2 = -(e^2 - 1)a^2 = -b^2 \text{ [by putting } (e^2 - 1)a^2 = b^2].$$

Therefore $$e^2 - 1 = \frac{b^2}{a^2},$$

which in (1) becomes

$$y^2 = \frac{b^2}{a^2}(2ax + x^2). \qquad (2)$$

102. Transform (2) of Art. 101, Cor., to C (Fig. 81).

The formulæ for this transformation become

$$x = x' - a, \qquad y = y',$$

which in (2) gives, after suppressing accents,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2), \qquad (1)$$

or $$a^2y^2 - b^2x^2 = -a^2b^2, \qquad (2)$$

which may be more symmetrically written,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad (3)$$

a form analogous to that of the equation of a right line (Art. 22, II).

Cor. 1.—By definition, $AF = e \cdot AO$, and $OA = \dfrac{p}{1+e}$ (Art. 100, Cor.); therefore,

$$AF = \frac{ep}{1+e} = \frac{e-1}{e^2-1}ep = a(e-1) \text{ [Art. 100, Cor.]};$$

that is, $$AF = a(e-1). \qquad (4)$$

$$OA = \frac{p}{1+e} = \frac{ep}{e^2-1} \cdot \frac{(e-1)}{e}$$

$$= \frac{a(e-1)}{e} \text{ [Art. 100, Cor.]};$$

that is, $$OA = \frac{a(e-1)}{e}. \tag{5}$$

$$CF = CA + AF = a + a(e-1) = ae;$$

that is, $$CF = ae. \tag{6}$$

$$CO = CA - AO = a - \frac{a(e-1)}{e} = \frac{a}{e};$$

that is, $$CO = \frac{a}{e}. \tag{7}$$

$$OF = p = CF - CO = ae - \frac{a}{e};$$

that is, $$OF = p = \frac{a(e^2-1)}{e}. \tag{8}$$

Cor. 2.—When $y = 0$, $x = \pm a$, which shows that the curve cuts the axis of x at two points equally distant from the origin, and on opposite sides of it. When $x = 0$, $y = \pm b\sqrt{-1}$; hence the curve cuts the axis of y in two *imaginary* points on opposite sides of the origin. We may, however, take two points B and B′, on different sides of C, making $CB = CB' = b$, as we shall have occasion to use them hereafter.

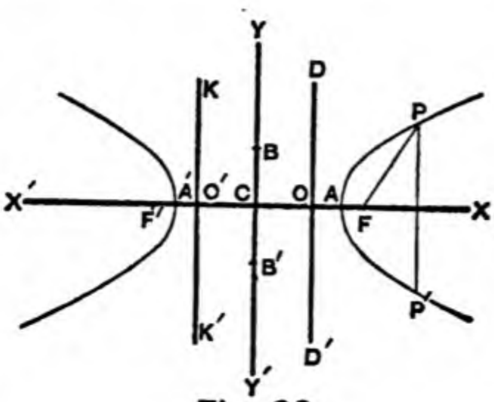

Fig. 82.

Cor. 3.—Solving (1) for y, we get

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2},$$

which shows that, for every value of $x > +a$ or $< -a$, there are two *real* values of y, numerically equal, with contrary signs; hence, for every point P on one side of the axis of x, there is a point P′ on the other side of the axis, at the same distance from it; and therefore the curve is symmetrical with respect to the axis of x. When x is $+a$ or $-a$, $y = \pm 0$; and for every value of x between $+a$ and $-a$, the two values of y are imaginary; therefore the curve is limited towards the centre by two tangents at A and A′.

Similarly, solving (1) for x, we get

$$x = \pm \frac{a}{b}\sqrt{y^2 + b^2},$$

which shows that for every value of y from $-\infty$ to $+\infty$ there are two real values of x, numerically equal with contrary signs; hence the curve is symmetrical with respect to the axis of y, and is unlimited in the direction of this axis.

SCH.—Because the curve is symmetrical with respect to the line BB′, it follows that if we take CF′ = CF (Fig. 82), and CO′ = CO, and draw KK′ perpendicular to OO′, the point F′ and the line KK′ will form respectively a second focus and directrix.

AA′ is called the **Transverse** axis of the hyperbola; BB′ is called the **Conjugate** axis of the hyperbola. In the *ellipse*, the conjugate axis is always less than the transverse axis (see Art. 70, Cor.), and therefore the former was called the *minor* and the latter the *major* axis. In the *hyperbola*, the conjugate axis may be greater than the transverse, since $b^2 = a^2(e^2 - 1)$ (Art. 101, Cor.), and e is > 1; therefore we do not call the axes in the hyperbola the *major* and *minor* axes.

The ratio e (Art. 99) is called the **Eccentricity** of the hyperbola.

The point C is called the **Centre** of the hyperbola, because it bisects every chord of the hyperbola which passes through it. This may be shown in the same way as in the case of the ellipse (Art. 71, Sch.).

COR. 4.—*To find the latus rectum* (Art. 53, Cor. 3).

Make $x = CF = ae$ (Cor. 1); denote the corresponding value of y by p; we have from Eq. (1) (Art. 102),

$$p^2 = \frac{b^2}{a^2}(a^2e^2 - a^2) = b^2(e^2 - 1) = \frac{b^4}{a^2} \text{ (Art. 101, Cor.).}$$

Therefore, $$2p = \frac{2b^2}{a} = \frac{4b^2}{2a} = \textit{latus rectum.}$$

Forming a proportion from this equation, we have

$$2a : 2b :: 2b : 2p.$$

That is, *the latus rectum is a third proportional to the tranverse axis and the conjugate.*

Since $b^2 = (e^2 - 1)a^2$ (Art. 101, Cor.), we have

$$a^2 + b^2 = a^2e^2;$$

that is, $$a^2 + b^2 = \overline{CF}^2 \text{ (Art. 102, Cor. 1).}$$

But $$a^2 + b^2 = \overline{AB}^2 \text{ (see Fig. 82).}$$

Therefore, $$AB = CF.$$

Hence, *the conjugate axis of the hyperbola is a perpendicular to the transverse axis at its centre, and is limited by an arc described with the vertex of the transverse axis as a centre, and with a radius equal to the distance from the focus to the centre.*

COR. 5.—Comparing equation (2), Art. 102, with (2) of Art. 71, we see that the equation of the hyperbola may be derived from that of the ellipse, by changing $+b^2$ into $-b^2$. Hence, we infer that *any function of b, expressing a property of the ellipse, will be converted into one expressing a corresponding property of the hyperbola, by changing b into* $b\sqrt{-1}$; therefore, in obtaining the properties of the hyperbola that are similar to those which have been proved for the ellipse, we shall, in most cases, either change the sign of b^2, or else refer the student to the corresponding demonstration in the ellipse.

By a process similar to that of Art. 71, Cor. 5, the details of which the student must supply, we obtain

$$y'^2 : y''^2 :: (x' + a)(x' - a) : (x'' + a)(x'' - a);$$

that is, *the squares of any two ordinates to the transverse axis of an hyperbola are to each other as the rectangles of the segments into which they divide the transverse axis.*

COR. 6.—A point is *outside*, *on*, or *inside* the hyperbola, according as $a^2y^2 - b^2x^2 + a^2b^2 >, =,$ or < 0. The proof is similar to that given in Art. 71, Cor. 6, for the ellipse.

A point is said to be *outside* the hyperbola if it lies in the space between the branches, so that no right line can be drawn through it to a focus without cutting the curve.

103. *To find the distance of any point in the hyperbola from the focus, in terms of the abscissa of the point.*

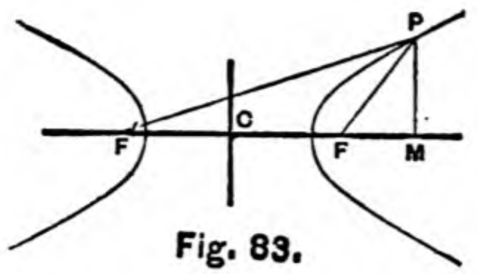

Fig. 83.

From the figure we have

$$\overline{FP}^2 = (x - ae)^2 + y^2$$
$$= (x - ae)^2 + \frac{b^2}{a^2}x^2 - b^2.$$
(Art. 102.)
$$= a^2 - 2aex + e^2x^2 \text{ (since } a^2e^2 - b^2 = a^2);$$

therefore, $\quad FP = ex - a.$

[We take only the *positive* value of the root, for the reason given in Art. 72.]

In like manner we find, by writing $-ae$ for $+ae$,

$$\overline{F'P}^2 = (x + ae)^2 + y^2 = a^2 + 2aex + e^2x^2;$$

therefore, $\quad F'P = ex + a.$

Hence, $\quad F'P - FP = 2a;$

or, *the difference of the distances of any point in an hyperbola from the foci is equal to the transverse axis.*

COR.—This result furnishes two other methods of constructing an hyperbola, having given the axes.

I. With C as a centre and BA as a radius, describe an arc cutting AA′ produced at F and F′; these points are the foci (Art. 102, Cor. 4). Now, with F′ as a centre and a radius greater than F′A, describe an arc; then with F as a centre, and a radius equal to that used before, diminished by the transverse axis AA′, describe another arc cutting the first at the point P; this will be a point of the curve, since

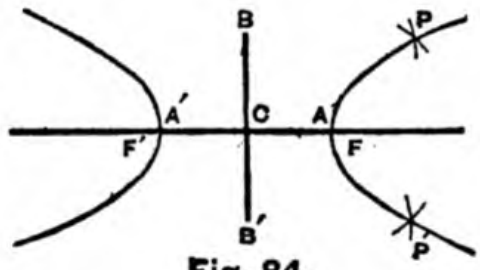

Fig. 84.

$$FP = F'P - 2a,$$

or
$$F'P - FP = 2a.$$

In the same way, any number of points may be found; joining these points, it will be a branch of the required hyperbola. By using F for the first centre and F′ for the second, with the same distances as before, any number of points of the other branch may be found.

II. Take a ruler, and fasten one end of it at F′, so it can revolve about F′ as a centre. Take a string whose length is less than that of the ruler by AA′, and fasten one end of it at F and the other end at B, the end of the ruler; then press the string against the edge of the ruler with the point of a pencil P, and revolve the ruler about F′, keeping the string tight; the pencil will describe one branch of an hyperbola, since, in every position of it, we shall have

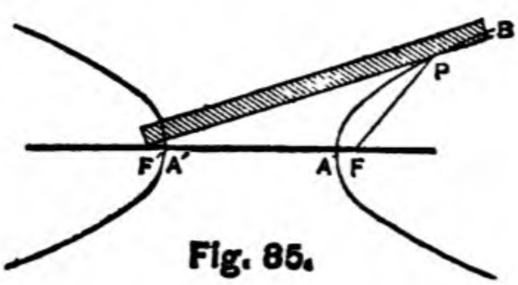

Fig. 85.

$$F'P - FP = AA'.$$

104. A Conjugate Hyperbola is one having the conjugate axis of a given hyperbola for its transverse axis, and the transverse axis of the given hyperbola for its conjugate axis. Either of two hyperbolas thus related is conjugate to the other. Thus, the hyperbola whose transverse axis is BB′ (Fig. 86) is the conjugate of the hyperbola whose transverse axis is AA′, and *conversely*, the latter is the conjugate of the former. They are often distinguished as the ***x*** **Hyperbola** and the ***y*** **Hyperbola**, each taking the name of the coordinate axis upon which its transverse axis lies; and when spoken of together are called **Conjugate Hyperbolas.**

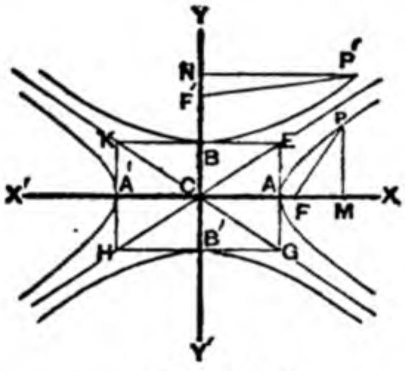

Fig. 86

105. *To find the equation of an hyperbola conjugate to a given hyperbola.*

By Art. 102, the equation of the given hyperbola is

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2),$$

or, Fig. 86,
$$\overline{PM}^2 = \frac{\overline{CB}^2}{\overline{CA}^2}(\overline{CM}^2 - \overline{CA}^2).$$

Hence, since P′ is a point on the conjugate hyperbola, having BB′ for its transverse axis and AA′ for its conjugate axis, we have,

$$\overline{NP'}^2 = \frac{\overline{CA}^2}{\overline{CB}^2}(\overline{CN}^2 - \overline{CB}^2),$$

or
$$x^2 = \frac{a^2}{b^2}(y^2 - b^2), \quad (1)$$

which is the equation of the conjugate hyperbola, and is the same expression we would obtain from the equation of the given hyperbola by putting $-b^2$ for $+b^2$, and $-a^2$ for $+a^2$.

SCH. 1.—Since the second hyperbola holds the same relation to the axis of y that the first does to the axis of x, we might have deduced the equation of the y hyperbola at once by changing a to b and b to a, x to y and y to x in the equation of the x hyperbola.

The sides of the rectangle described on the axes are the tangents to the four branches at the vertices.

Comparing the equation of the conjugate hyperbola with that of the given hyperbola, we have (Art. 101, Cor.),

$$(e^2 - 1)\, b^2 = a^2,$$

or

$$a^2 + b^2 = b^2e^2 = \overline{CF'}^2.$$

(See Art. 102, Cor. 1.) Therefore the foci of the y hyperbola are at the same distance from the centre as the foci of the x hyperbola, but the *eccentricity* of the former has a different value from that of the latter.

SCH. 2.—The equations of the diagonals CE and CG are respectively

$$y = \frac{b}{a}x \qquad \text{and} \qquad y = -\frac{b}{a}x.$$

If in the equations of the two conjugate hyperbolas we make $b = a$, we have (Art. 102),

$$y^2 - x^2 = -a^2, \tag{2}$$

and (1) of the present Art. becomes

$$y^2 - x^2 = a^2. \tag{3}$$

These hyperbolas are called **Equilateral** hyperbolas, from the equality of the axes. The equilateral hyperbola corresponds to the case in which the ellipse becomes a circle. (See Art. 71, Cor. 7.) The peculiarity in the figure of the equilateral hyperbola is that *the curve is identical in form with its conjugate.* From Art. 101, Cor., we have

$$e^2 - 1 = \frac{b^2}{a^2};$$

therefore, in the *equilateral* hyperbola we have $e = \sqrt{2}$.

106. *To construct a pair of conjugate hyperbolas whose axes are given.*

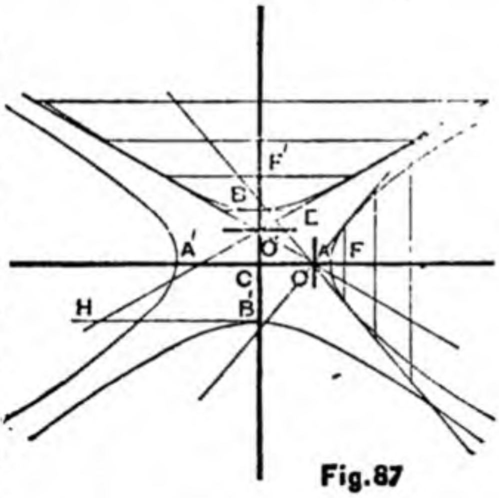

Fig. 87

Draw the axes AA' and BB' at right angles to each other; construct the x hyperbola as in Art. 99. Now take $CF' = CF$, which equals AB (Art. 105, Sch.), and F' is the focus of the y hyperbola. Take $BE = BF'$, and $B'H = B'F'$; draw through E and H a right line; it is one of the focal tangents. Through O' draw a line perpendicular to BB'; this is the directrix corresponding to the focus F' of the y hyperbola. The construction is now the same as in Art. 99.

107. *To find the equation of the tangent at any point of an hyperbola.*

To obtain this equation for the hyperbola, we change b^2 into $-b^2$ in equations (6), (7), and (11) of Art. 74, and get

$$a^2yy' - b^2xx' = -a^2b^2, \tag{1}$$

$$y = \frac{b^2x'}{a^2y'}x - \frac{b^2}{y'}, \tag{2}$$

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{3}$$

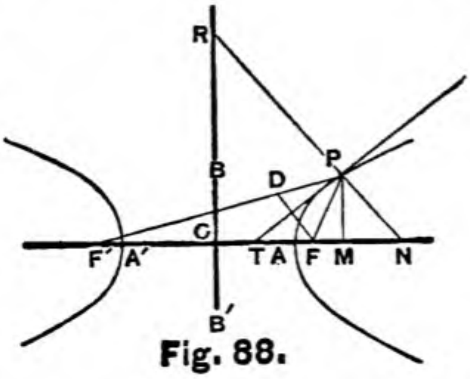

Fig. 88.

Cor.—To find the point in which the tangent cuts the axis of x, make $y = 0$ in (1), and get

$$x = \frac{a^2}{x'} = CT,$$

which is the same value we found for the abscissa of the point at which the tangent cuts the axis of x in the ellipse.

(Art. 74, Cor. 2). This value of x has the same sign as x'; hence, for the right-hand branch, it is always positive; that is, the tangent to the right-hand branch cuts the axis of x to the right of the centre.

By Art. 102, Cor. 2, we have $F'C = FC = ae$; therefore we have

$$F'T = ae + \frac{a^2}{x'} = \frac{a}{x'}(ex' + a),$$

and
$$FT = ae - \frac{a^2}{x'} = \frac{a}{x'}(ex' - a).$$

Hence, $F'T : FT :: ex' + a : ex' - a :: F'P : FP$

(by Art. 103). That is, *the tangent of an hyperbola cuts the distance between the foci in segments proportional to the adjacent focal radii of contact; and therefore it bisects the internal angle between these focal radii.*

This principle affords a method of drawing a tangent to an hyperbola at a given point.

Let P be the given point (see Fig. 88). Draw the focal radii F′P and FP to the given point P. On the longer, F′P, lay off PD = PF, and join DF. Through P draw PT perpendicular to DF; PT will be the trangent required, for it bisects the angle FPF′.

The *subtangent* $MT = CM - CT = x' - \frac{a^2}{x'}$. That is,

$$\textit{the subtangent} = \frac{x'^2 - a^2}{x'}.$$

108. *To find the equation of the normal at any point of an hyperbola.*

We change b^2 into $-b^2$ in (2) of Art. 75, and get

$$y - y' = -\frac{a^2y'}{b^2x'}(x - x'), \qquad (1)$$

which is the required equation of the normal at (x', y').

COR. 1.—To find the point in which the normal cuts the axis of x, we make $y = 0$ in (1), and get, after reduction

$$x = \frac{a^2 + b^2}{a^2} x' = \text{CN (Fig. 88)} = e^2 x' \text{ (Art. 101, Cor.)}.$$

The subnormal $\text{MN} = \text{CN} - \text{CM}$

$$= \frac{a^2 + b^2}{a^2} x' - x' \ = \frac{b^2}{a^2} x'.$$

SCH.—The expression $\text{CN} = e^2 x'$ enables us, as in the case of the ellipse (Art. 75, Sch.), to draw a normal at any point P of the hyperbola, or one from any point N of the transverse axis.

COR. 2.—By Art. 102, Cor. 1,

$$\text{F}'\text{C} = \text{FC} = ae;$$

therefore we have

$$\text{F}'\text{N} = e\,(ex' + a),$$

and

$$\text{FN} = e\,(ex' - a).$$

Hence, $\text{F}'\text{N} : \text{FN} :: ex' + a : ex' - a :: \text{F}'\text{P} : \text{FP}$

(Art. 103). That is, *the normal of an hyperbola cuts the distance between the foci in segments proportional to the adjacent focal radii of contact; and hence it bisects the external angle between the focal radii of contact.*

109. *To find the locus of the intersection of the tangent at any point with the perpendicular on it from either focus.*

Changing the sign of b^2 in (3) and (4) of Art. 76, and adding the squares of the resulting equations together, we get

$$x^2 + y^2 = a^2,$$

for the required locus, which is therefore a circle described on the transverse axis.

110. *To find the co-ordinates of the point of contact of a tangent to an hyperbola from a fixed point.*

Let (x', y') be the required point of contact, and (x'', y'') the fixed point through which the tangent passes.

Changing $+b^2$ to $-b^2$ in the results of Art. 77, we get

$$x' = \frac{a^2b^2x'' \mp a^2y''\sqrt{a^2y''^2 - b^2x''^2 + a^2b^2}}{b^2x''^2 - a^2y''^2},$$

$$y' = \frac{a^2b^2y'' \mp b^2x''\sqrt{a^2y''^2 - b^2x''^2 + a^2b^2}}{b^2x''^2 - a^2y''^2}.$$

These values indicate that from any fixed point *two* tangents can be drawn to an hyperbola, *real, coincident;* or *imaginary,* according as

$$a^2y''^2 - b^2x''^2 + a^2b^2 >, =, \text{ or } < 0;$$

that is, according as the point (x'', y'') is *outside, on,* or *inside* the curve (Art. 102, Cor. 6).

COR.—It is clear that if any two real tangents be drawn from a given point to touch the *same branch,* their abscissas of contact will have *like* signs; and *unlike,* if they touch *different* branches. Hence, since the values of x in the former case must have the same signs, we have, regarding only their *numerical* values,

$$a^2b^2x'' > a^2y''\sqrt{a^2y''^2 - b^2x''^2 + a^2b^2};$$

or squaring, transposing, and reducing, we have

$$y'' < \frac{b}{a}x''. \qquad (1)$$

But (Art. 105, Sch. 2) $y = \frac{b}{a}x$ is the equation of the diagonal of the rectangle formed upon the axes of the hyperbola; therefore, the ordinate of the point from which two tangents can be drawn to the *same branch* of an hyperbola must be less than the corresponding ordinate of the diagonal; that is, the point itself must be somewhere between the diagonals (CE, CG) or (CH, CK) produced, and the adjacent branch of the curve (Fig. 86). These diagonals produced are called **Asymptotes** of the hyperbola, which we shall consider in Art. 113. Hence, generally, the two tangents which can be drawn to an hyperbola from any external point, will both touch the *same branch*, if the external point be between that branch and the adjacent portions of the asymptotes; but if the external point be so placed that we cannot pass from it to the curve without crossing an asymptote, the two tangents touch *different branches* of the curve.

111. *Tangents are drawn to an hyperbola from a given external point; to find the equation of the chord of contact* (Art. 77).

Change b^2 into $-b^2$ in (5) of Art. 78, and get

$$a^2yy' - b^2xx' = -a^2b^2, \qquad (1)$$

which is the equation of the chord of contact.

112. *Through any fixed point a chord is drawn to an hyperbola, and tangents to the hyperbola are drawn at the extremities of the chord; to find the equation of the locus of the intersection of the tangents, when the chord is turned about the fixed point.*

Change b^2 into $-b^2$ in (3) of Art. 79, and get

$$a^2yy' - b^2xx' = -a^2b^2, \qquad (1)$$

which is the equation required, and the locus is a right line.

SCH.—The line (1) is called the **Polar** of the point (x', y') with regard to the hyperbola $a^2y^2 - b^2x^2 = -a^2b^2$, and the point (x', y') is called the **Pole** of the line.

The statements in Art. 49 with respect to the circle may all be applied to the hyperbola as they were to the parabola (Art. 61), and the same conclusions arrived at that were reached in Arts. 49 and 61, and referred to in the ellipse (Art. 79, Sch.).

113. An **Asymptote** of a curve is a line which continually approaches the curve, and becomes tangent to it only at an infinite distance, while it passes within a finite distance of the origin. We have called the diagonals produced of the rectangle on the axes (Art. 110, Cor.), the *asymptotes* of the hyperbola; we now proceed to show that they are such, that is, that they meet the curve only at infinity.

114. *To prove that the diagonals of the rectangle on the axes are asymptotes to both the given and conjugate hyperbolas.*

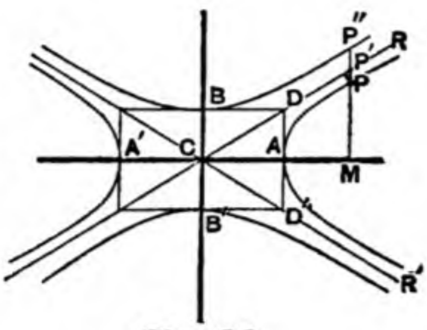

Fig. 89.

Produce the ordinate MP of any point P in the given hyperbola, to meet the diagonal CR and the conjugate hyperbola, in the points P′ and P″ respectively. The distance of the point P from CR = PP′ sin PP′C, and therefore it varies as PP′. Now, if CM, the common abscissa $= x$, PM $= y$, P′M $= y'$, and P″M $= y''$, we have, from the equations of the given hyperbola, the diagonal, and the conjugate hyperbola,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2), \tag{1}$$

$$y'^2 = \frac{b^2}{a^2}x^2, \tag{2}$$

$$y''^2 = \frac{b^2}{a^2}(x^2 + a^2). \tag{3}$$

Subtracting (1) from (2), we have

$$y'^2 - y^2 = b^2, \quad \text{or} \quad y' - y = \frac{b^2}{y + y'}. \tag{4}$$

Subtracting (2) from (3), we have

$$y''^2 - y'^2 = b^2, \quad \text{or} \quad y'' - y' = \frac{b^2}{y'' + y'}. \tag{5}$$

If now we suppose the abscissa CM to increase continually, and the line MP to move parallel to itself, the ordinates y, y', and y'' will increase continually, and therefore, from (4) and (5), $y' - y$ and $y'' - y'$ will diminish continually; and when x (CM), and therefore y, y', and y'' become infinitely great, $y' - y$ and $y'' - y'$ will become infinitely small; that is, as x increases indefinitely, the two curves continually approach the diagonal CR, and become tangent to it and to each other only at infinity. Hence the diagonals are asymptotes to both curves.

COR. 1.—The equations of CR and CR′ are (Art. 105, Sch. 2),

$$y = \frac{b}{a}x \qquad \text{or} \qquad \frac{x}{a} - \frac{y}{b} = 0;$$

and

$$y = -\frac{b}{a}x \qquad \text{or} \qquad \frac{x}{a} + \frac{y}{b} = 0;$$

therefore the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ includes both asymptotes.

COR. 2.—Let $ACR = \theta$, $ACR' = \theta'$; then

$$\tan\theta = \frac{b}{a}, \qquad \tan\theta' = -\frac{b}{a};$$

$$\sin\theta = \frac{b}{\sqrt{a^2 + b^2}}, \qquad \cos\theta = \frac{a}{\sqrt{a^2 + b^2}} = \frac{1}{e};$$

$$\sin\theta' = -\frac{b}{\sqrt{a^2 + b^2}}, \qquad \cos\theta' = \frac{a}{\sqrt{a^2 + b^2}} = \frac{1}{e}.$$

115. *To find the equation of any diameter.* (Def. of Art. 62.)

Change b^2 into $-b^2$ in (2) of Art. 80, and get

$$y = \frac{b^2}{a^2}\cot\theta \cdot x \qquad (1)$$

for the required equation.

Since a^2 and b^2 are constant for any given hyperbola, and θ is constant for any given system of parallel chords, (1) is the equation of a right line passing through the origin, that is, through the centre of the hyperbola. Hence, every diameter of the hyperbola passes through the centre. By giving to θ a suitable value, (1) may be made to represent *any* right line passing through the centre. Hence, *every* right line that passes through the centre of an hyperbola is a diameter; that is, it bisects some system of parallel chords.

SCH.—To draw a diameter of an hyperbola, draw any two parallel chords, and bisect them; the line passing through the points of bisection is a diameter. The intersection of two diameters will be the centre of the hyperbola.

COR. 1.—Let θ' = the inclination of the diameter itself to the transverse axis; then we have

$$\tan\theta' = \frac{y}{x};$$

which in (1) gives

$$\tan\theta \tan\theta' = \frac{b^2}{a^2},$$

as the relation between θ and θ' when they are the angles which a system of parallel chords and their diameter respectively make with the axis of x.

COR. 2.—Writing the equation of the diameter in the form

$$y = \tan\theta \cdot x, \qquad (1)$$

and eliminating y between this equation and that of the given hyperbola, to find the abscissas of the points of intersection of (1) and the curve, we obtain

$$x = \pm \frac{ab}{\sqrt{b^2 - a^2 \tan^2\theta}}. \qquad (2)$$

Now, eliminating y between (1) and the equation of the conjugate hyperbola (Art. 105), to find the abscissas of the points of intersection of (1) and the conjugate curve, we obtain

$$x = \pm \frac{ab}{\sqrt{a^2 \tan^2 \theta - b^2}}. \qquad (3)$$

If $a^2 \tan^2 \theta < b^2$, that is, if $\tan \theta < \pm \frac{b}{a}$, the values of x in (2) are *real*, showing that (1) intersects the given hyperbola at finite distances from the centre; while the values of x in (3) are *imaginary*, showing that (1) does not cut the y hyperbola.

If $a^2 \tan^2 \theta > b^2$, that is, if $\tan \theta > \pm \frac{b}{a}$, the values of x in (2) are *imaginary*, showing that (1) does not cut the given hyperbola; while the values of x in (3) are *real*, showing that (1) cuts the y hyperbola at finite distances from the centre.

If $a^2 \tan^2 \theta = b^2$, that is, if $\tan \theta = \pm \frac{b}{a}$, the values of x in (2) and (3) are infinite, showing that (1) does not cut either the x or the y hyperbola. In this case, (1) coincides with the diagonals of the rectangle described on the axes of the two conjugate hyperbolas (Art. 105, Sch. 2), that is, with the asymptotes (Art. 113).

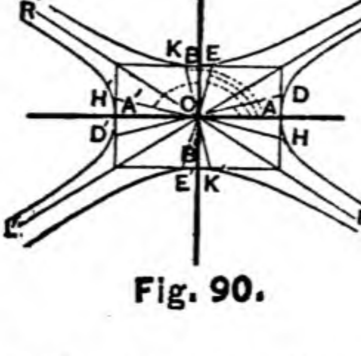

Fig. 90.

We learn, then, that diameters which cut the given hyperbola in real points, must either make with the transverse axis an angle *less* than is made by the first of these diagonals, or *greater* than is made by the second, as DD′ and HH′. If they cut the *conjugate* hyperbola in real points, they must either make with the transverse axis an angle *greater* than is made by the first of these diagonals, or *less* than is made by the second, as EE′ and KK′. If they *coincide* with these diagonals, as LL′ and RR′, they will intersect the hyperbolas at an infinite distance. Hence, every right line drawn through the centre of an hyperbola must meet the hyperbola or its conjugate, unless it coincides with one of the asymptotes.

116. *If one diameter of an hyperbola bisects all chords parallel to a second diameter, the second will bisect all chords parallel to the first.*

Let θ and θ' be the respective inclinations of any two diameters to the transverse axis. Then the condition that the first diameter shall bisect all chords parallel to the second diameter (Art. 115, Cor. 1) is

$$\tan\theta \tan\theta' = \frac{b^2}{a^2}. \qquad (1)$$

But this is also the condition that the second diameter bisects all chords parallel to the first.

SCH.—Two diameters are **Conjugate** when each bisects all chords parallel to the other.

Because the conjugate of any diameter is parallel to the chords which that diameter bisects, therefore the inclinations of two conjugates must be connected in the same way as those of a diameter and its bisected chords. Hence, if θ and θ' are the inclinations, the *equation of condition for conjugate diameters* in the hyperbola (Art. 115, Cor. 1) is

$$\tan\theta \tan\theta' = \frac{b^2}{a^2}. \qquad (2)$$

This condition shows that the tangents of inclination of any two conjugate diameters have *like* signs; therefore it indicates that the angles made with the transverse axis by the two conjugates are either *both acute* or *both obtuse.* Therefore, *conjugate diameters of an hyperbola lie on the same side of the conjugate axis,* as CD and CE, or CK and CH′ (see Fig. 90).

COR.—From (2), if

$$\tan\theta < \frac{b}{a}, \qquad \tan\theta' > \frac{b}{a};$$

and if $$\tan\theta > -\frac{b}{a} \qquad \tan\theta' < -\frac{b}{a}.$$

Therefore (Art. 115, Cor. 2), *if one of two conjugates*, DD′, *meets an hyperbola, the other*, EE′, *meets the conjugate hyperbola.*

117. *The tangent at either extremity of any diameter is parallel to its conjugate diameter.*

[For demonstration, see Art. 82.]

118. *Given the co-ordinates* x', y' *of one extremity of a diameter, to find the co-ordinates* x'', y'' *of either extremity of the conjugate diameter.*

By the **Extremities** of the conjugate diameter, we mean the points in which the conjugate cuts the conjugate hyperbola.

Let (x', y') be the point D (Fig. 90), and (x'', y'') the point E or E′. Since the conjugate diameter EE′ is parallel to the tangent at (x', y') (Art. 117), and passes through the origin, therefore its equation (Art. 107) is

$$y = \frac{b^2x'}{a^2y'}x;$$

which, combined with the equation of the conjugate hyperbola (Art. 105) gives

$$x'' = \pm\frac{a}{b}y'; \qquad y'' = \pm\frac{b}{a}x'.$$

We see that the upper signs of the co-ordinates are both positive and the lower signs both negative, while in the ellipse (Art. 83), the upper signs are unlike and the lower also. This agrees with the properties of the two curves developed in Arts. 81 and 116, Sch.

119. *To express the length of a semi-diameter* (**a′**), *and its conjugate* (**b′**), *in terms of the abscissa of the extremity of the diameter.*

Let (x', y') and (x'', y'') be the extremities D and E of the diameters DD′ and EE′; then we have

$$a'^2 = x'^2 + y'^2 = x'^2 + \frac{b^2}{a^2}(x'^2 - a^2) \text{ (Art. 102)}$$

$$= \frac{a^2 + b^2}{a^2}x'^2 - b^2;$$

therefore $$a'^2 = e^2x'^2 - b^2 \text{ (Art. 101, Cor.).} \quad (1)$$

Also, $$b'^2 = x''^2 + y''^2 = \frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2 \text{ (Art. 118)}$$

$$= x'^2 - a^2 + \frac{b^2}{a^2}x'^2 \text{ (Art. 102);}$$

therefore, $$b'^2 = e^2x'^2 - a^2 \text{ (Art. 101, Cor.).} \quad (2)$$

Cor. (1) — (2) gives

$$a'^2 - b'^2 = a^2 - b^2;$$

that is, the difference of the squares of any two conjugate diameters of an hyperbola is equal to the difference of the squares of the axes.

120. *To find the length of the perpendicular from the centre to the tangent at any point.*

Let (x', y') be the point, and p the perpendicular. The equation of the tangent at (x', y') is (Art. 107),

$$a^2yy' - b^2xx' = -a^2b^2. \quad (1)$$

Therefore (Art. 24),

$$p = \frac{a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{ab}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}} = \frac{ab}{b'}. \text{ (Art. 119.)}$$

121. *To find the angle between any pair of conjugate diameters.*

Let ϕ be the required angle ECD in Fig. 90. By the same process as in the ellipse (Art. 86), we find

$$\sin\phi = \frac{ab}{a'b'}. \qquad (1)$$

COR.—Clearing (1) of fractions, we have

$$a'b'\sin\phi = ab, \qquad (2)$$

which shows that *the area of the parallelogram whose sides touch the hyperbola at the ends of any pair of conjugate diameters is constant, and equal to the rectangle of the axes.*

SCH.—By Art. 119, Cor., $a'^2 - b'^2 =$ a constant; therefore, a' and b' increase or decrease together; hence, by causing D to move along the hyperbola from A, E also will move along from B (Fig. 90). But any diameter CD tends towards an *infinite length*, as its inclination tends towards the limit $\theta = \tan^{-1}\frac{b}{a}$ (Art. 115, Cor. 2); therefore its semi-conjugate CE tends towards infinity; and, as $a'b'\sin\phi$ is *constant*, and a' and b' tend towards infinity, $\sin\phi$ tends towards 0; or, the angle between two conjugates of an hyperbola diminishes without limit. When the two conjugates approach infinity in length, they tend to coincide with the diagonals of the rectangle constructed on the axes; but they are never equal, since $a'^2 - b'^2$ is always equal to $a^2 - b^2$ (Art. 119, Cor.), unless the curve is equilateral. Therefore, the infinite diameters which form the limit of the conjugates are not *equal* infinites, and hence we do not, as in the ellipse, have *equi-conjugates*. We may, however, call these conjugates in their limit, when they coincide with each other and with either of the asymptotes, **Self Conjugates**,* since each is a diameter conjugate to itself.

* See Howison's Analytic Geometry, p. 381.

The inclinations of the self-conjugate diameters to the transverse axis are determined by the equation

$$\tan\theta = \pm\frac{b}{a}. \quad \text{(Art. 115, Cor. 2.)}$$

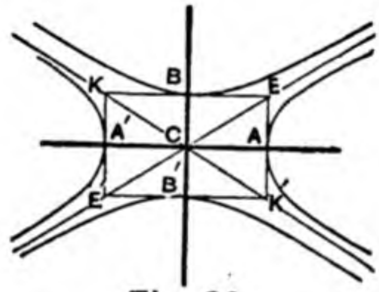

Fig. 91.

The first value corresponds to the angle ACE, and the second value to the angle ACK (Fig. 91).

The inclination of these self-conjugates to each other, as ECK or ECK′, is determined by

$$\sin\phi = 2\sin BCE\cos BCE$$
$$= 2\frac{a}{\sqrt{a^2+b^2}}\times\frac{b}{\sqrt{a^2+b^2}};$$

that is, $$\sin\phi = \frac{2ab}{a^2+b^2},$$

where $$\phi = ECK \text{ or } ECK'.$$

122. *If a chord and diameter of an hyperbola are parallel, the supplemental chord and the conjugate diameter are parallel.* (See Def., Art. 88.)

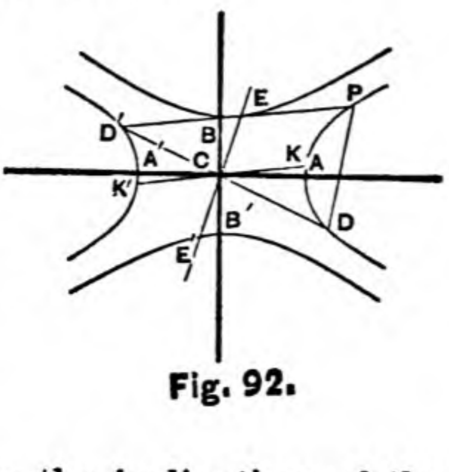

Fig. 92.

Let DD′ be a diameter of the hyperbola; PD and PD′ two supplemental chords, the first parallel to the diameter EE′; then will the supplemental chord PD′ be parallel to the conjugate diameter KK′.

Let (x', y') be the point D, and therefore $(-x', -y')$ will be the point D′. Let ϕ and ϕ' be the inclinations of the two chords DP and D′P. Then, by the same process as in Art. 89, or simply by changing b^2 into $-b^2$ in that Art., we get

$$\tan\phi\tan\phi' = \frac{b^2}{a^2},$$

as the condition that the two chords DP and D′P shall be supplemental.

Now, from Art. 116, Sch., we have

$$\tan\theta\tan\theta' = \frac{b^2}{a^2},$$

as the condition that two diameters shall be conjugate to each other; the rest of the argument of Art. 89 applies directly to the hyperbola. Therefore, the supplemental chord PD′ is parallel to the conjugate diameter KK′.

123. *To find the equation of the hyperbola referred to any pair of conjugate diameters.*

To do this we must transform the equation of the hyperbola

$$a^2y^2 - b^2x^2 = -a^2b^2, \tag{1}$$

from rectangular to oblique axes, having the same origin.

Let DD′ and SS′ be two conjugate diameters. Take CD for the new axis of x, and CS for the new axis of y. Denote the angle ACD by θ and ACS by θ'. Let x, y be the co-ordinates of any point P of the hyperbola referred to the old axes, and x', y' the co-ordinates of the same point referred to the new axes.

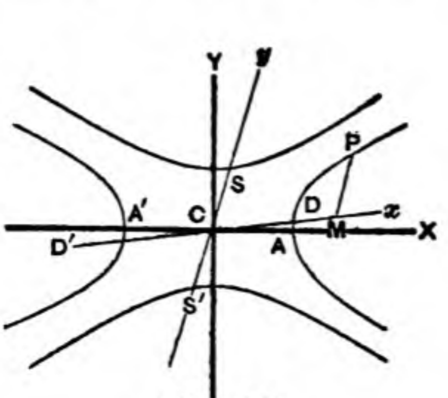

Fig. 93.

Now we may use the same process employed in Art 90; or, we may simply change b^2 into $-b^2$ in (3) of Art. 90 (see Art. 102, Cor. 5), and get

$$(a^2\sin^2\theta - b^2\cos^2\theta)\,x'^2 + (a^2\sin^2\theta' - b^2\cos^2\theta')\,y'^2 = -a^2b^2. \tag{1}$$

Let a' and b' denote the lengths of the semi-diameters CD and CS. If we make $y'=0$ in (1), we get

$$x'^2 = \frac{-a^2b^2}{a^2\sin^2\theta - b^2\cos^2\theta} = a'^2. \qquad (2)$$

Also, if in (1) we make $x'=0$, we get

$$y'^2 = \frac{-a^2b^2}{a^2\sin^2\theta' - b^2\cos^2\theta'} = -b'^2. \qquad (3)$$

We put this latter equal to $-b'^2$, because we have supposed the new axis of x to meet the given hyperbola, as in Fig. 93; therefore we know (Art. 116, Cor.) that the new axis of y will *not* meet the given hyperbola; hence $\frac{-a^2b^2}{a^2\sin^2\theta' - b^2\cos^2\theta'}$ is a negative quantity.

From (2) we get $\quad a^2\sin^2\theta - b^2\cos^2\theta = -\frac{a^2b^2}{a'^2}. \qquad (4)$

From (3) we get $\quad a^2\sin^2\theta' - b^2\cos^2\theta' = \frac{a^2b^2}{b'^2}. \qquad (5)$

Substitute (4) and (5) in (1), divide by $-a^2b^2$, omit accents from the variables, and we get

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1, \qquad (6)$$

or,

$$a'^2y^2 - b'^2x^2 = -a'^2b'^2, \qquad (7)$$

which is the equation required, and is of the same form as when referred to the axes of the curve (Art. 102).

Similarly, the equation of the *conjugate* hyperbola referred to the same pair of conjugate diameters is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = -1, \qquad (8)$$

or

$$a'^2y^2 - b'^2x^2 = a'^2b'^2. \qquad (9)$$

[Let the student give the demonstration.]

COR.—Comparing (7) with (9) of Art. 90, we see that the equation of the hyperbola may be derived from that of the ellipse by changing b'^2 into $-b'^2$. Hence, we infer that *any function of b' expressing a property of the ellipse will be converted into one expressing a corresponding property of the hyperbola by changing b' into $b'\sqrt{-1}$.*

124. *To find the equation of a tangent to the hyperbola referred to any pair of conjugate diameters.*

By reasoning exactly as in Art. 91, using the term "hyperbola" for "ellipse," or, by changing b'^2 into $-b'^2$ in (1) of Art. 91, according to Art. 123, Cor., we get

$$a'^2yy' - b'^2xx' = -a'^2b'^2, \tag{1}$$

which is the required equation.

COR.—To find where the tangent cuts the axis of x, make $y = 0$ in (1), and get

$$x = \frac{a'^2}{x'}.$$

125. *To prove that tangents at the extremities of any chord of an hyperbola meet on the diameter which bisects that chord.*

Take the diameter CD, which bisects the chord PP', for the axis of x, and the conjugate diameter CS for the axis of y.

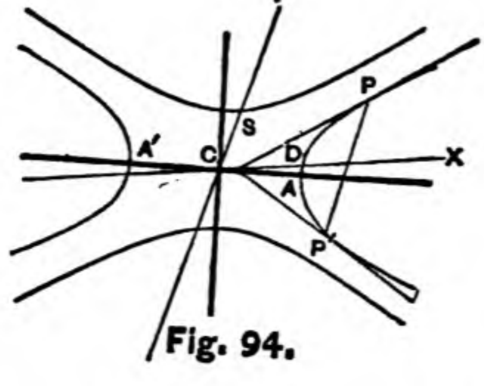

Fig. 94.

Now reason as in Art. 92, or change b'^2 into $-b'^2$ in (1) and (2) of Art. 92, according to Art. 123, Cor., and get

$$a'^2yy' - b'^2xx' = -a'^2b'^2, \tag{1}$$

and
$$-a'^2yy' - b'^2xx' = -a'^2b'^2, \tag{2}$$

which are the equations of the tangents at the extremities of the chord PP′ referred to the diameter CD which bisects PP′, and the conjugate diameter CS. Now, by Art. 124, Cor., both of these tangents cut the axis of x at the point $\left(\frac{a'^2}{x'}, 0\right)$, which proves the proposition.

126. *If tangents are drawn at the extremities of any focal chord of an hyperbola:*

I. *The tangents will intersect on the corresponding directrix.*

II. *The line drawn from the point of intersection of the tangents to the focus will be perpendicular to the focal chord.*

I. Reasoning as in Art. 93, we find for the equation of the chord of contact (Art. 111),

$$a^2yy' - b^2xx' = -a^2b^2, \tag{1}$$

which, for the right-hand focus $(ae, 0)$, becomes

$$-b^2aex' = -a^2b^2;$$

or

$$x' = \frac{a}{e}; \tag{2}$$

that is, the point of intersection of the tangents is on the corresponding directrix (Art. 102, Cor. 1), showing that the directrix is the polar of the focus. (Art. 79, Sch.)

II. The equation of the line passing through the right-hand focus and the point (x', y') is, by (Art. 26),

$$y = \frac{y'}{x' - ae}(x - ae). \tag{3}$$

From (2), $x' = \frac{a}{e}$, which in (3) gives

$$y = \frac{y'e}{a - ae^2}(x - ae)$$

$$= -\frac{aey'}{b^2}(x - ae). \text{ (Art. 101, Cor.)} \tag{4}$$

The equation of the chord of contact [see (1) above] is

$$y = \frac{b^2x'}{a^2y'}x - \frac{b^2}{y'};$$

which becomes, for the focal chord [since $x' = \frac{a}{e}$, from (2)],

$$y = \frac{b^2}{aey'}x - \frac{b^2}{y'}, \qquad (5)$$

which is perpendicular to (4), by Art. 27, Cor. 1.

127. *Find the locus of the point of intersection of two tangents to an hyperbola at right angles to each other.*

Reason as in Art. 94, or change b^2 into $-b^2$ in equation (5) of that Art., and get

$$x^2 + y^2 = a^2 - b^2 \qquad (1)$$

as the required locus. Hence, the locus is a circle with its centre at C, and with $\sqrt{a^2 - b^2}$ for its radius, *unless* $b^2 > a^2$, in which case the locus is impossible; that is, two tangents cannot be drawn at right angles to each other when b^2 is greater than a^2.

128. *The rectangle of the focal perpendiculars upon any tangent is constant, and equal to the square of the semi-conjugate axis.*

Call p and p' the perpendiculars. The equation of the tangent at any point (x', y') is

$$a^2yy' - b^2xx' = -a^2b^2.$$

By Art. 24,

$$p = +\frac{b^2x'ae - a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{b\,(ex' - a)}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}}$$

$$= \frac{b}{b'}(ex' - a). \quad \text{(Art. 119.)}$$

Also, $$p' = -\frac{-b^2x'ae - a^2b^2}{\sqrt{a^4y'^2 + b^4x'^2}} = \frac{b\,(ex' + a)}{\sqrt{\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2}};$$

$$= \frac{b}{b'}(ex' + a). \quad \text{(Art. 119.)}$$

Hence, $$pp' = \frac{b^2}{b'^2}(e^2x'^2 - a^2) = b^2. \quad \text{(Art. 119.)}$$

129. *To find the polar equation of the hyperbola, the left focus being the pole.*

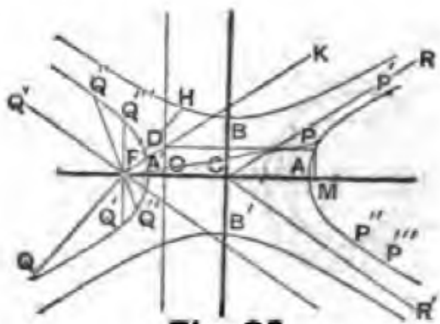

Fig. 95.

Let $F'P = r$; $AF'P = \theta$; then (Def. of Art. 99) we have,

$$F'P = e \cdot PD$$
$$= e\,(F'M - F'O)$$
$$= e \cdot MF' - e \cdot F'O$$
$$= e \cdot F'P \cos AF'P - a(e^2 - 1) \quad \text{(Art. 102, Cor. 1)},$$

or $$r = er\cos\theta - a\,(e^2 - 1);$$

therefore, $$r = \frac{a\,(e^2 - 1)}{e\cos\theta - 1}, \qquad (1)$$

which is the equation required.

Cor.—When $\theta = 0$, $r = ae + a = F'C + CA = F'A$, as it should do. (Art. 102, Cor. 1)

When $e\cos\theta - 1 = 0$, that is, when $\theta = \cos^{-1}\frac{1}{e}$,

$$r = \frac{a\,(e^2 - 1)}{0} = \infty.$$

But in this case r, or F'K, is parallel to the asymptote CR (See Art. 114, Cor, 2, and Fig. 89). That is, while θ increases from 0 to $\cos^{-1}\frac{1}{e}$, r increases from $ae + a$ to ∞, tracing the branch APP'.

When θ passes the value $\cos^{-1}\frac{1}{e}$, $e\cos\theta - 1$ becomes negative, and therefore r becomes negative, and the left-hand branch is generated, the negative end of r tracing QQ′Q″; thus, when $\theta = \text{AF}'\text{H}$, r, being negative, is reckoned backwards to Q.

When $\theta = 90°$, $r = -a(e^2-1) = -\frac{b^2}{a}$ (Art. 101, Cor.), which equals the *semi latus rectum*, p, with a negative sign (Art. 102, Cor. 4), and Q′ is located.

When $\theta = 180°$, $r = -a(e-1) = a - ae = -\text{F}'\text{A}'$, as it should.

While θ increases from 90° to 270°, the arc Q′Q″A′Q‴ is traced with the negative end of r.

When $\theta = 270°$, $r = -a(e^2-1) = -\frac{b^2}{a} = -p$, and the point Q‴ is *located.*

While θ increases from 270° to $\cos^{-1}\frac{1}{e}$, r remains negative, and increases numerically from p to ∞, its negative end tracing Q‴, Q$^{\text{iv}}$.

At $\theta = \cos^{-1}\frac{1}{e}$ in the fourth quadrant, $r = \infty$, and is parallel to the asymptote CR′.

While θ increases from $\cos^{-1}\frac{1}{e}$ to 360°, r is positive, and diminishes from ∞ to $a + ae$, and the arc P‴, P″, A is traced.

130. *To find the polar equation of the hyperbola when the pole is at the centre.*

Changing $a^2y^2 - b^2x^2 = -a^2b^2$ into a system of polar co-ordinates (as in Art. 97), we have

$$r^2 = \frac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta} = \frac{b^2}{e^2\cos^2\theta - 1} \text{ (Art. 101, Cor.) (1)}$$

Similarly, the polar equation of the conjugate hyperbola is

$$r^2 = \frac{b^2}{1 - e^2 \cos^2 \theta}. \tag{2}$$

COR.—Equation (1) shows that for every value of θ between $-\cos^{-1}\frac{1}{e}$ and $+\cos^{-1}\frac{1}{e}$, r has two *real* values, numerically equal, with contrary signs. These two values of r, taken together, make the diameter; hence, every diameter of the hyperbola is bisected at the centre (Art. 102, Sch.).

When $\theta = \cos^{-1}\frac{1}{e}$, $r = \infty$; but in this case, r or CR, Fig. 89, coincides with the asymptote.

While θ increases from $\cos^{-1}\frac{1}{e}$ to $\left(180^\circ - \cos^{-1}\frac{1}{e}\right)$, r is imaginary, showing that it does not reach either branch of the given hyperbola.

Equation (2) shows that for every value of θ between $\cos^{-1}\frac{1}{e}$ and $\left(180^\circ - \cos^{-1}\frac{1}{e}\right)$, r has two real values, numerically equal, with contrary signs. These two values of r, taken together, make the diameter of the conjugate hyperbola; hence, every diameter of the conjugate hyperbola is bisected at the centre.

When $\theta = -\cos^{-1}\frac{1}{e}$, $r = \infty$; in this case, r coincides with the asymptote.

For every value of θ between $-\cos^{-1}\frac{1}{e}$ and $+\cos^{-1}\frac{1}{e}$, r is imaginary, showing that it does not reach either branch of the given hyperbola.

In (1), r is least when $\theta = 0$, giving

$$r = \sqrt{\frac{b^2}{e^2 - 1}},$$

which equals a (Art. 101, Cor.). In (2), r is least when $\theta = 90°$, giving $r = b$. Hence, in the hyperbola, each axis is the minimum diameter of its own curve.

Also, it is evident from both (1) and (2) that the value of r is the same for θ and $(\pi - \theta)$. Therefore, diameters which make supplemental angles with the transverse axis of an hyperbola are equal.

131. The properties of the hyperbola hitherto established are similar to those of the ellipse. We have now to consider some properties *peculiar to the hyperbola,* arising from the presence of the *asymptotes.* (See Art. 113.)

132. *To prove that the asymptotes are the diagonals of every parallelogram formed on a pair of conjugate diameters.*

The equations of the hyperbola and its asymptotes, when referred to the axes of the curve, are respectively

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad (1)$$

and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \qquad (2)$$

When we transform the equation of the hyperbola to its conjugate diameters (Art. 123), equation (1) becomes

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1;$$

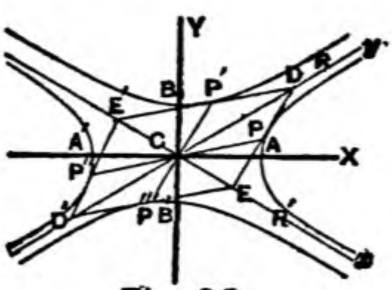

Fig. 96.

therefore we may at once infer that (2) transformed to the same conjugate diameters, becomes

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0;$$

that is, the equations of the asymptotes CR and CR′, referred to any pair of conjugate diameters, are

$$\frac{x}{a'} - \frac{y}{b'} = 0, \qquad (3)$$

and

$$\frac{x}{a'} + \frac{y}{b'} = 0. \qquad (4)$$

Take $CP = a'$, and $CP' = b'$.

Equation (3), or $y = \frac{b'}{a'}x$, is the equation of a line passing through the origin and the point (a', b') (see Art. 26, Cor. 4), that is, through C and D; and (4), or $y = -\frac{b'}{a'}x$, is a line passing through the origin and $(a', -b')$, that is, through C and E. Hence, (3) and (4), which are the asymptotes, are also the diagonals of the parallelogram EDE′D′ on the conjugate diameters PP″ and P′P‴.

133. *To find the equation of the hyperbola referred to its asymptotes as axes.*

To do this, we must transform the equation

$$a^2y^2 - b^2x^2 = -a^2b^2 \qquad (1)$$

from rectangular to oblique axes, having the same origin.

Let CX and CY be the old axes (Fig. 96). Take the lower asymptote CR′ for the new axis of x, and the other, CR, for the new axis of y.

Let x, y be the co-ordinates of any point P in the curve referred to the old axes, and x', y' the co-ordinates of the same point referred to the new axes. Denote the angles ACR′ and ACR by θ and θ' respectively.

The formulæ for transformation (Art. 35, Cor. 1) are

$$x = x' \cos\theta + y' \cos\theta',$$
$$y = x' \sin\theta + y' \sin\theta'.$$

Squaring, substituting in (1), and arranging, we have

$$\left\{\begin{array}{r} (a^2 \sin^2\theta - b^2 \cos^2\theta)\, x'^2 \\ + (a^2 \sin^2\theta' - b^2 \cos^2\theta')\, y'^2 \\ + 2\,(a^2 \sin\theta \sin\theta' - b^2 \cos\theta \cos\theta')\, x'y' \end{array}\right\} = -a^2b^2. \qquad (2)$$

From Art. 114, Cor. 2, we have

$$\tan^2\theta = \frac{b^2}{a^2} = \tan^2\theta';$$

from which we get

$$a^2\sin^2\theta - b^2\cos^2\theta = 0, \tag{3}$$

and

$$a^2\sin^2\theta' - b^2\cos^2\theta' = 0. \tag{4}$$

Also, from Art. 114, Cor. 2, we have

$$\sin\theta\sin\theta' = -\frac{b^2}{a^2+b^2},$$

and

$$\cos\theta\cos\theta' = \frac{a^2}{a^2+b^2};$$

therefore,

$$a^2\sin\theta\sin\theta' - b^2\cos\theta\cos\theta' = -\frac{2a^2b^2}{a^2+b^2}. \tag{5}$$

Substituting (3), (4), and (5) in (2), we get

$$-\frac{4a^2b^2}{a^2+b^2}x'y' = -a^2b^2;$$

or suppressing accents from the variables and reducing, we have

$$xy = \frac{a^2+b^2}{4}, \tag{6}$$

and putting m^2 for $\frac{a^2+b^2}{4}$, we have,

$$xy = m^2, \tag{7}$$

which is the equation required.

COR.—The equation of the *conjugate* hyperbola, referred to the same axes, is (Art. 105)

$$xy = -m^2. \tag{8}$$

If we solve (7) for x, we get

$$x = \frac{m^2}{y},$$

which shows that as y increases, x diminishes, and when $y = \infty$, $x = 0$; that is, the curve approaches the axis of y, and finally touches it at an infinite distance from the centre.

Similarly, the curve approaches the axis of x, and finally touches it at an infinite distance from the centre.

SCH.—The second member of (7) is essentially positive, and of (8) essentially negative; hence, both x and y have the same sign in (7) and contrary signs in (8); therefore one branch of the given hyperbola lies wholly in the first angle and the other in the third; while one branch of the *conjugate* hyperbola lies wholly in the second and the other in the fourth angle. (See Fig. 96.)

In the case of *equilateral* hyperbolas (Art. 105, Sch. 2), the angle between the asymptotes, which (Art. 121, Sch.) is equal to $\sin^{-1}\dfrac{2ab}{a^2 + b^2} = \sin^{-1} 1$, becomes a right angle; therefore, the *equilateral* hyperbola is also called the **Rectangular** hyperbola.

134. *To find the equation of the tangent at any point of an hyperbola referred to the asymptotes as axes.*

Let (x', y') and (x'', y'') be any two points, P and P′, on the curve. The equation of the secant through these points (Art. 26), is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'). \quad (1)$$

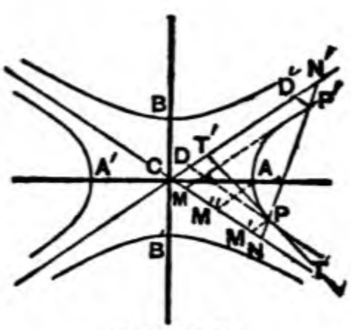

Fig. 97.

Since (x', y') and (x'', y'') are on the curve, we have (Art. 133),

$$x'y' = m^2 = x''y'', \qquad \text{or} \qquad y'' = \frac{x'y'}{x''},$$

which in (1) gives $\quad y - y' = -\dfrac{y'}{x''}(x - x'), \quad$ (2)

which is the equation of the *secant* to the hyperbola.

When the points become consecutive, we have $x'' = x'$; hence (2) becomes

$$y - y' = -\frac{y'}{x'}(x - x'). \qquad (3)$$

Clearing (3) of fractions, transposing, and uniting, we have

$$x'y + y'x = 2x'y',$$

or

$$\frac{x}{x'} + \frac{y}{y'} = 2, \qquad (4)$$

which is the equation of the tangent required.

COR. 1.—Making y and x successively $= 0$ in (4) we get

$$x = 2x' = \text{CT}, \qquad \text{and} \qquad y = 2y' = \text{CT}'.$$

Hence, P is the middle point of TT′; therefore, *the portion of the tangent included between the asymptotes is bisected at the point of contact.*

COR. 2.—From Cor. 1, we have,

$$\text{CT} \times \text{CT}' = 4x'y' = a^2 + b^2 \text{ (Art. 133).}$$

That is, *the rectangle of the intercepts cut off upon the asymptotes by any tangent is constant, and equal to the sum of the squares on the semi-axes.*

COR. 3.—The area of the triangle TCT′, Fig. 97, is

$$= \tfrac{1}{2}\text{CT} \times \text{CT}' \sin \text{TCT}'$$

$$= 2x'y' \times \frac{2ab}{a^2 + b^2} \text{ (Cor. 1, and Art. 121, Sch.),}$$

$$= \frac{a^2 + b^2}{2} \times \frac{2ab}{a^2 + b^2} \text{ (Art. 133)}$$

$$= ab = \textit{constant.}$$

Therefore, *the triangle included between any tangent and the asymptotes is constant, and equal to the rectangle of the semi-axes.*

135. *To prove that the intercepts of a secant between the hyperbola and its asymptotes are equal.*

In equation (2) of Art. 134, make $y = 0$, and get

$$x = x'' + x'$$
$$= \text{CN (Fig. 97).}$$

Hence, $\text{CN} - x' = x''$,

or $\text{M'N} = \text{D'P'}$;

therefore, $\text{NP} = \text{N'P'}$;

that is, *the intercepts of the secant are equal.*

SCH.—This proposition affords a convenient method of constructing the curve. If the axes are given, construct the rectangle on them, the diagonals of which are the asymptotes. Then through the extremity of the transverse axis, draw a right line intercepted by the asymptotes; lay off on this line from one asymptote a distance equal to the extremity of the axis from the other asymptote; the point thus found will be a point of the curve. In this manner, find any number of points, and draw a line through them; *this will be the required curve.*

136. *To prove that the parallelogram formed by drawing lines from any point of an hyperbola parallel to and terminating in the asymptotes, is equal to one-eighth the rectangle on the axes.*

Call ϕ the angle TCT' (Fig. 97); the area of CM'PD

$$= x'y' \sin \phi$$
$$= \frac{a^2 + b^2}{4} \times \frac{2ab}{a^2 + b^2} \text{ (Art. 133, and Art. 121, Sch.)}$$
$$= \tfrac{1}{2}ab = \tfrac{1}{8}(2a \cdot 2b),$$

which proves the proposition.

137. *To find the equations of two conjugate diameters of an hyperbola referred to its asymptotes.*

The diameter which passes through the origin and the point P (x', y') is represented (see Art. 26, Cor. 4) by

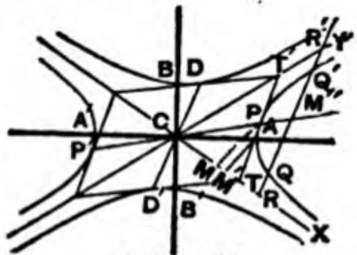

Fig. 98.

$$y = \frac{y'}{x'} x,$$

or
$$\frac{x}{x'} - \frac{y}{y'} = 0. \qquad (1)$$

The diameter conjugate to this one, CD, is parallel to the tangent at (x', y'), and therefore (Art. 134, Eq. 4) its equation is

$$y = -\frac{y'}{x'} x,$$

or
$$\frac{x}{x'} + \frac{y}{y'} = 0. \qquad (2)$$

COR.—When the diameters PP′ and DD′ become the axes, AA′ and BB′, we have, since the axes bisect the angle between the asymptotes,

$$CM' = M'A, \qquad \text{or} \qquad x' = y';$$

therefore (1) and (2) become

$$x - y = 0, \qquad \text{and} \qquad x + y = 0,$$

which are the equations of the axes referred to the asymptotes.

138. *Given the co-ordinates of the extremity of a diameter, to find those of the extremity of its conjugate.*

Let (x', y') be the point P (Fig. 98), and (x'', y'') the point D.

The equation of DD′ (Art. 137) is

$$\frac{x}{x'} + \frac{y}{y'} = 0. \qquad (1)$$

The equation of the conjugate hyperbola (Art. 133) is

$$xy = -m^2. \qquad (2)$$

Eliminating between (1) and (2), we get

$$x'' = \mp x', \qquad y'' = \pm y'. \qquad (3)$$

COR. 1.—The equation of the tangent at P (x', y') (Art. 134) is

$$\frac{x}{x'} + \frac{y}{y'} = 2. \qquad (4)$$

The equation of the tangent at D (x'', y''), the extremity of the conjugate diameter (Art. 134) is

$$\frac{x}{x''} + \frac{y}{y''} = 2,$$

or from (3),

$$\frac{x}{x'} - \frac{y}{y'} = -2. \qquad (5)$$

Adding (4) and (5), we get $x = 0$ as the locus of the intersection of the tangents (4) and (5), which is the equation of the axis of y, or the asymptote CR′. Therefore, *tangents at the extremities of conjugate diameters meet on the asymptotes.*

COR. 2.—Since T′ is a vertex of the parallelogram formed on the conjugate diameters PP′ and DD′, we have

$$PT' = CD;$$

therefore,

$$TT' = 2PT' = DD';$$

that is, *the portion of the tangent at any point of an hyperbola, included between the asymptotes, is equal to the diameter conjugate to that which passes through the point of contact.*

139. *If a chord be drawn parallel to any diameter, it will be bisected by the conjugate diameter produced.*

Let QQ′ be drawn parallel to DD′ (Fig. 98); then will it be bisected at M″ by CP produced.

Since QQ′ is parallel to DD′, its equation will differ from that of DD′ only by a constant term; therefore [Art. 138, (1)]

$$\frac{x}{x'} + \frac{y}{y'} = c \tag{1}$$

is the equation of QQ′.

Combine (1) with the equation of PP′ (Art. 137), which is

$$\frac{x}{x'} - \frac{y}{y'} = 0, \tag{2}$$

and we get $\qquad x = \frac{1}{2}cx', \qquad y = \frac{1}{2}cy',$

as the co-ordinates of M″. But from (1) we have

$$\text{CR} = cx', \qquad \text{and} \qquad \text{CR}' = cy';$$

therefore M″ is the middle point of RR′. But (Art. 135),

$$\text{RQ} = \text{R}'\text{Q}';$$

therefore, QM″ = M″Q′, *which proves the proposition.*

EXAMPLES.

1. Find the axes of the hyperbola whose equation is $3y^2 - 2x^2 + 12 = 0$; also the eccentricity of the given and the conjugate hyperbola, and the parameter.

Ans. $a = \sqrt{6},\ b = 2;\ e = \sqrt{\frac{5}{3}};\ e' = \sqrt{\frac{5}{2}};\ 2p = \frac{8}{\sqrt{6}}.$

2. Find the intersection of the hyperbola $3y^2 - 2x^2 + 12 = 0$ and the circle $x^2 + y^2 = 16$. *Ans.* $(\pm 2\sqrt{3},\ \pm 2)$.

3. Find whether the line $y = \frac{3}{4}x$ cuts the hyperbola $5y^2 - 2x^2 = -15$, or its conjugate.

Ans. It cuts the conjugate.

4. Find the equation of an hyperbola of given transverse axis, whose vertex bisects the distance between the centre and the focus. *Ans.* $y^2 - 3x^2 = -3a^2$.

5. If the ordinate MP (Fig. 95) of an hyperbola be produced to Q, so that $MQ = F'P$, find the locus of Q.

Ans. A right line.

6. If an ellipse and an hyperbola have the same foci, prove that their tangents at the points of intersection are at right angles. (See Art. 75, Cor. 2, and Art. 107, Cor.)

7. Find the condition that the line $\left(\frac{x}{m} + \frac{y}{n} = 1\right)$ shall touch the hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$. *Ans.* $\frac{a^2}{m^2} - \frac{b^2}{n^2} = 1$.

[To obtain this, compare $\frac{x}{m} + \frac{y}{n} = 1$ with equation of tangent (Art. 107), which is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1,$$

and we have $\qquad \frac{x'}{a} = \frac{a}{m}$ and $\frac{y'}{b} = -\frac{b}{n}$,

which in equation of curve gives the answer.]

8. Find where the tangents from the foot of the directrix will meet the hyperbola, and what angle they will make with the transverse axis.

Ans. The extremity of the *latus rectum;* $\tan^{-1} \pm e$.

9. Find the angle included between the asymptotes of the hyperbola $16y^2 - 9x^2 = -25$. *Ans.* $73° 44'$.

10. Find the perpendicular from the focus of any hyperbola to its asymptotes. *Ans.* The semi-conjugate axis.

11. If $3AC = 2CF'$ (Fig. 95), find the inclination of the asymptotes to the transverse axis.

Ans. $\tan^{-1} \frac{\sqrt{5}}{2}$.

12. If the asymptotes of the hyperbola are axes, show that the equation of one directrix is $x + y - a = 0$.

[See Art. 137, Cor.]

13. Prove that if a circle be described with the focus of an hyperbola for its centre and with a radius equal to the semi-conjugate axis, it will touch the asymptotes in the points where they are cut by the directrix.

14. Prove that the radius of a circle which touches an hyperbola and its asymptotes is equal to that part of the latus rectum produced which is intercepted between the curve and the asymptote.

15. Find the length of the normal NP and of RP (Fig. 88). [See Art. 108, Cor. 1.] *Ans.* $\text{NP} = \frac{bb'}{a}$, $\text{RP} = \frac{ab'}{b}$.

16. Prove that the product of the two perpendiculars let fall from any point of an hyperbola upon the asymptotes is constant and equal to $\frac{a^2b^2}{a^2 + b^2}$.

17. Tangents to an hyperbola are drawn from any point on either branch of the conjugate curve; prove that their chord of contact touches the opposite branch of the conjugate curve.

[Take the diameter passing through the *point* for axis of y, and the conjugate diameter for axis of x; equation of chord of contact is

$$\frac{xx'}{a'^2} - \frac{yy'}{b'^2} = 1,$$

which soon reduces to $y = \pm b'$; $\therefore$ etc.]

18. In any equilateral hyperbola, let $\phi =$ the inclination of a diameter, passing through any point P, and $\phi' =$ that of the polar of P, the transverse axis being the axis of x; prove that $\tan\phi \tan\varphi' = 1$.

[Equation of diameter is $y = \frac{y'}{x'}x$; $\therefore$ $\frac{y'}{x'} = \tan\phi$; polar of P is $xx' - yy' = a^2$; $\therefore$ $\frac{x'}{y'} = \tan\phi'$; $\therefore$ etc.]